Math 8 Homework 1

Directions: Homework should be nicely written with attention to presentation and readability. Remember, somebody is reading your work!

1 Logical Statements

(a) Write the following statement using logical connectives ($\exists$, $\Rightarrow$, etc).

For any subset S of the real numbers, S is compact if and only if it is closed and bounded.

What can we deduce about a set S which is not compact?

(b) A real-valued function f defined on the real numbers is called *uniformly continuous* if and only if

For every $\epsilon > 0$ there is a $\delta > 0$ so that for any real x, y which satisfy $|x - y| < \delta$, it follows that $|f(x) - f(y)| < \epsilon$.

Rewrite this definition using logical connectives and give a precise definition of what it means for a function not to be uniformly continuous.

(c) Lagrange's theorem says that if H is a subgroup of G then the size of H divides the size of G. Which of the following are generally true?

(i) If H is a subgroup of G and has size 14, then the size of G is even.

(ii) If G is has size 32, then any subgroup has an even size.

(iii) If G has size 12 then there is a subgroup H of size 6.

2 Direct and Contradiction Proof

(a) Let n be an integer. Prove that n is even if and only if n^2 is even.

(b) Prove that $\sqrt{2}$ is irrational.

(c) Prove or disprove: the sum of two irrationals is irrational.

(d) Prove there exist irrational numbers x, y so that x^y is *rational.* Hint: start by thinking about $\sqrt{2}$; oddly enough, your final proof probably won't identify what the numbers x and y are!

(e) The following proof that $\sqrt{6} + \sqrt{2} < 4$ is invalid:

Squaring both sides of $\sqrt{6} + \sqrt{2} < 4$ gives $6 + 2\sqrt{12} + 2 < 16$, which rearranges into $\sqrt{12} < 4$. Squaring this gives $12 < 16$, which is true.

Explain why this proof is invalid and provide a correct proof.

3 Quantifiers

In some of the following problems, you might want to use this theorem from calculus:

Theorem (Intermediate Value)**.** *Let f be a continuous function on $[a, b]$. Suppose that $x, y \in [a, b]$ are given, with $f(x) \leq f(y)$. Then given any z so that $f(x) \leq z \leq f(y)$ there is a point c between x and y so that $f(c) = z$.*

(a) Prove there is a number c in $(0, 1)$ so that $(1 - c)\cos c = \sin c$.

(b) A continuous function f satisfies $f(0) = f(2)$. Prove there is a number c for which $f(c + 1) = f(c)$.

(c) Prove that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that whenever $n > m \geq N$ it follows that $1/m - 1/n < \epsilon$.

(d) Disprove the following: for every positive integer n, the integer $n^2 + n + 41$ is prime.

(e) Let f be a differentiable function on $[0, 4]$. Prove there is a point c for which $f'(c) < 1 + f(c)^2$.